MATH2050C Selected Solution to Assignment 6

Section 3.4

(4a). The subsequence $b_n = a_{2n} = 1/(2n) \to 0$ as $n \to \infty$. On the other hand, the subsequence $c_n = a_{2n+1} = 2 + 1/(2n+1) \to 2$ as $n \to \infty$. Since these two subsequences converge to different limits, $\{a_n\}$ is divergent.

(b). The subsequence $b_k = a_{8k} = \sin 8k\pi/4 = 0$ while the subsequence $c_k = a_{8k+2} = \sin(8k+2)\pi/4 = 1$. Thus the first subsequence tends to 0 and the second one to 1. We conclude that this sequence is divergent.

(7a). Observe $a_n = (1+1/n^2)^{n^2}$ is a subsequence of $c_n = (1+1/n)^n$. In fact, $a_n = c_{n^2}$. Since every subsequence converges to the same limit for a convergent sequence, we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = e$.

(d). In a previous exercise we have shown that $a_n = (1+2/n)^n$ is convergent (actually when 2 is replaced by any positive a). Denote its limit by a. Then the subsequence $b_k = a_{2k} = (1+1/k)^{2k}$ should tend to the same a. But now it is clear that it converges to e^2, so $a = e^2$. Therefore, $\lim_{n\to\infty} a_n = e^2$.

Alternatively, for $x_n = (1+2/n)^n$, $x_{2n} = (1+2/2n)^{2n} \to e^2$. We have

$$\Big(1+\frac{1}{2n+2}\Big)^{2n+2}\Big(1+\frac{1}{2n+2}\Big)^{-1} \le \Big(1+\frac{1}{2n+1}\Big)^{2n+1} \le (1+\frac{1}{2n}\Big)^{2n}\Big(1+\frac{1}{2n}\Big).$$

By Squeeze Theorem we conclude $(1+1/(2n+1))^{2n+1} \to e^2$. Since both the even and odd subsequences converge to e^2, the entire sequence converges to e^2.

(8a) Write $(3n)^{1/2n}$ as $(3n)^{1/3n\times 3/2}$. As $n^{1/n} \to 1$, $(3n)^{1/3n} \to 1$, so $(3n)^{1/2n} \to 1^{3/2} = 1$. Recall that $x_n \to x$ implies $x_n^a \to x^a$ for positive numbers.

(9). If $\{x_n\}$ does not converge to 0, for some $\varepsilon_0 > 0$, there are $n_j \to \infty$ such that $|x_{n_j} - 0| \ge \varepsilon_0$. Thus every convergent subsequence of $\{x_{n_j}\}$ cannot converge to 0.

(11). Let $a_n = (-1)^n x_n$. By assumption it tends to some a. The subsequence $b_k = a_{2k} = x_{2k}$ tends to a, showing that $a \ge 0$. On the other hand, $c_k = a_{2k+1} = -x_{2k+1}$ also tends to a, showing that $a \le 0$. (Recall it is assumed that all $x_n \ge 0$.) We conclude that $a = 0$. For every $\varepsilon > 0$, there is some n_ε such that $|x_n - 0| = |(-1)^n x_n - 0| < \varepsilon$ for all $n \ge n_\varepsilon$, hence $\{x_n\}$ converges to 0.

Supplementary Problems

1. Let $\{x_n\}$ be a positive sequence such that $a = \lim_{n\to\infty} x_{n+1}/x_n$ exists. Show that $\lim_{n\to\infty} x_n^{1/n}$ exists and is equal to a.

   Solution Assume $a > 0$ first. For small ε satisfying $a - \varepsilon > 0, |x_n/x_{n-1} - a| < \varepsilon$ for all $n \ge n_0$. We have

$$0 < (a-\varepsilon)^{n-n_0} < \frac{x_n}{x_{n-1}}\frac{x_{n-1}}{x_{n-2}}\cdots\frac{x_{n_0+1}}{x_{n_0}} < (a+\varepsilon)^{n-n_0}.$$

That is,

$$(a-\varepsilon)^{n-n_0}x_{n_0} < x_n < (a+\varepsilon)^{n-n_0}x_{n_0} ,$$

or

$$(a-\varepsilon)^{1-n_0/n}x_{n_0}^{1/n} < x_n^{1/n} < (a+\varepsilon)^{1-n_0/n}x_{n_0}^{1/n} .$$

Let x_{n_k} be a subsequence with $x_{n_k}^{1/n_k}$ converging to b. Taking $x_n = x_{n_k}$ in the above estimate and passing to the limit, we get

$$a-\varepsilon \le b = \lim_{n_k\to\infty} x_{n_k}^{1/n_k} \le a+\varepsilon .$$

Since ε is arbitrary, $b=a$. We have shown that every convergent subsequence converges to the same number a, hence $x_n^{1/n}$ converges to a. When $a=0$, replace the term $(a-\varepsilon)^{n-n_0}$ on the left by 0 and follow the same argument.

2. Show that $\lim_{n\to\infty} \dfrac{n}{(n!)^{1/n}} = e$.

   Solution Let $x_n = n^n/n!$ so that $\dfrac{n}{(n!)^{1/n}} = x_n^{1/n}$. Now, $x_{n+1}/x_n = (1+1/n)^n \to e$ and the desired conclusion follows from the result in Problem 1.

   Remark A formula relating $n!$ to n^n is given by the Stirling's formula: $n! \sim \sqrt{2\pi n}(n/e)^n$.

3. The concept of a sequence extends naturally to points in $\mathbb{R}^N$. Taking $N=2$ as a typical case, a sequence of ordered pairs, $\{\mathbf{a}_n\}, \mathbf{a}_n = (x_n, y_n)$, is said to be convergent to $\mathbf{a}$ if, for each $\varepsilon > 0$, there is some n_0 such that

   $$|\mathbf{a}_n - \mathbf{a}| < \varepsilon , \quad \forall n \ge n_0 .$$

   Here $|\mathbf{a}| = \sqrt{x^2+y^2}$ for $\mathbf{a} = (x,y)$. Show that $\lim_{n\to\infty} \mathbf{a}_n = \mathbf{a}$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

   Solution It follows from the elementary inequalities

   $$|x_1-y_1|, |x_2-y_2| \le |\mathbf{a}-\mathbf{b}| \le |x_1-y_1| + |x_2-y_2| ,$$

   which show that $\mathbf{a}_n \to \mathbf{a}$ if and only if $x_n \to x$ and $y_n \to y$.

4. Bolzano-Weierstrass Theorem in $\mathbb{R}^N$ reads as, every bounded sequence in $\mathbb{R}^N$ has a convergent subsequence. Prove it. A sequence is bounded if $|\mathbf{a}_n| \le M$, $\forall n$, for some number M.

   Solution Take $N=2$ for simplicity. $\{\mathbf{a}_n\}$ is bounded implies $\{x_n\}$ and $\{y_n\}$ are bounded by the previous exercise. Pick a convergent subsequence $\{x_{n_k}\}$ from $\{x_n\}$. As $\{y_{n_k}\}$ is a bounded sequence, pick a convergent sequence $\{y_{n_{k_j}}\}$ from $\{y_{n_k}\}$. Then $(x_{n_{k_j}}, y_{n_{k_j}})$ is a convergent subsequence for $\mathbf{a}_n = (x_n, y_n)$.

5. Consider the sequence $\{x_n\}, x_n = \sum_{k=1}^n s_n 1/n^2$ where s_n is either 1 or -1. Show that $\{x_n\}$ is convergent.

   Solution We show that $\{x_n\}$ is Cauchy. We already know that the sequence $y_n = \sum_{k=1}^n 1/k^2$ converges and hence it is Cauchy. For $\varepsilon > 0$, $|y_m - y_n| = |\sum_{k=n+1}^m 1/k^2 = 1/(n+1)^2 + \cdots + 1/m^2| < \varepsilon$ for all $m, n \ge n_0$. Now, $|x_m - x_n| = |\sum_{k=n+1}^m s_k 1/k^2| \le |\sum_{k=n+1}^m 1/k^2| < \varepsilon$, $\forall m, n \ge n_0$. Therefore $\{x_n\}$ is also Cauchy and hence converges by the Cauchy Convergence Criterion.

6. Consider $x_n = (x_{n-1} + x_{n-2})/2, n \geq 3$ and $x_1 = 1, x_2 = 2$. Show that $\{x_n\}$ converges to $5/3$. Hint: To find the limit establish $x_{2n+1} = 1+\frac{1}{2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{2n-1}}$ and $x_{2n} = x_{2n-1}+\frac{1}{2^{2n-2}}$.

   Solution We have

$$|x_{n+1} - x_n| = \frac{1}{2}|x_n - x_{n-1}| = \cdots = \frac{1}{2^{n-1}} \ .$$

Hence

$$|x_m - x_n| \leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| = \frac{1}{2^{m-2}} + \frac{1}{2^{m-3}} + \cdots + \frac{1}{2^{n-1}} < \frac{1}{2^{n-2}} \ ,$$

which implies $\{x_n\}$ is a Cauchy sequence. By induction one can show that

$$x_{2n+1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{2n-1}}.$$

So

$$x_{2n+1} = 1 + \frac{2}{3}\left(1 - \frac{1}{4^n}\right) \to 5/3 \ .$$